ABSTRACT

Zero-Sum Magic Graphs and Their Null Sets

by

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For any $h \in \mathbb{N}$, a graph $G = (V, E)$, with vertex set V and edge set E, is said to be h-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_h - \{0\}$ such that the induced vertex labeling $l^+ : V(G) \to \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. When this constant is 0 we call G a zero-sum h-magic graph. The null set of G is the set of all natural numbers $h \in \mathbb{N}$ for which G admits a zero-sum h-magic labeling. A graph G is said to be uniformly null if every magic labeling of G induces zero sum. In this thesis we will identify the null sets of certain classes of Planar Graphs.

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

1.1 Magic-Labelings

In this thesis all graphs are connected, finite, simple, and undirected. For graph theory notations and terminology not directly defined in this thesis, we refer readers to [2]. For an abelian group A, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can induce a vertex set labeling $l^+ : V(G) \to A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be A-*magic* if there is a labeling $l : E(G) \to A - \{0\}$ such that for each vertex v, the induced vertex label is a constant map.

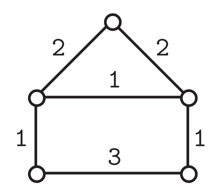

Figure 1.1. A $\mathbb{Z}$-magic graph.

In general, a graph G may have multiple labelings that show the graph is A-magic. For example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a magic labeling of G with sum c, then $l : E(G) \to A - \{0\}$, the *inverse labeling* of l, defined by $l(uv) = -l(uv)$ will be another magic labeling of G with sum $-c$. A graph $G = (V, E)$ is called *fully magic* if it is A-magic for every abelian group A. For example, every regular graph is fully magic. A graph $G = (V, E)$ is called *non-magic* if for every abelian group A, the graph is not A-magic. The most obvious example of a non-magic graph is P_n $(n \geq 3)$, the path of order n. As a result, any graph with a path pendant of length at least two would be non-magic. Here is another example of a non-magic graph: Consider the graph H in Figure 1.2. Given any abelian group A, a potential magic labeling of H is illustrated in that figure. The condition $l^+(u) = l^+(v)$ implies that $6x + y = 7x + y$ or $x = 0$, which is not an acceptable magic labeling. Thus H is not A-magic.

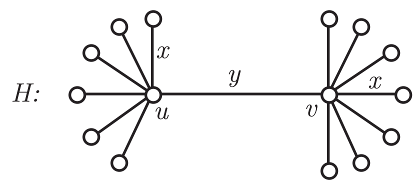

Figure 1.2. An example of a non-magic graph.

Certain classes of non-magic graphs are presented in [1].

The original concept of A-magic graph originated with J. Sedlacek [16, 17], who defined A-magic-graphs to be a graph with a real-valued edge labeling such that:

1. distinct edges have distinct nonnegative labels

2. the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [4] proved that a graph G is magic if and only if every edge of G is contained in a (1-2)-factor. $\mathbb{Z}$-magic graphs were considered by Stanley [18, 19], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [3, 20]. For convenience, the notation 1-magic will be used to indicate $\mathbb{Z}$-magic and $\mathbb{Z}_h$-magic graphs will be referred to as h-magic graphs. Clearly, if a graph is h-magic, it is not necessarily k-magic ($h \neq k$).

1.2 Integer-Magic Labelings

Definition 1.1. For a given graph G the set of all positive integers h for which G is h-magic is called the *integer-magic spectrum* of G and is denoted by $IM(G)$.

Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph H, Figure 1.2, is non-magic, therefore $IM(H) = \emptyset$. The integer-magic spectra of certain classes of graphs created through the amalgamation of cycles and stars have been identified in [6] and [7] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of certain other graphs have been studied in [5, 8, 9, 10, 13, 14, 15].

CHAPTER 2

ZERO-SUM MAGIC GRAPHS

2.1 Zero-Sum Magic

Definition 2.1. An h-magic graph G is said to be *h-zero-sum* (or just zero-sum) if there is a magic labeling of G in $\mathbb{Z}_h$ that induces a vertex labeling with sum 0. The graph G is said to be uniformly zero-sum if any magic labeling of G induces 0 sum.

A direct result of this definition is that any graph that has an edge pendant is not zero-sum.

Definition 2.2. The null set of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that G is h-magic and admits a zero-sum labeling in $\mathbb{Z}_h$.

Here are some well known results concerning null sets of graphs by E. Salehi in [11, 12]

Theorem 2.1. *If* $n \geq 4$*, then* $N(K_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is odd} \\ \mathbb{N} - \{2\} & \text{if } n \text{ is even} \end{cases}$

Theorem 2.2. *Let* $m, n \geq 2$*. Then*

$$N(K(m,n)) = \begin{cases} \mathbb{N} & \text{if } \ m+n \text{ is even;} \\ \mathbb{N} - \{2\} & \text{if } \ m+n \text{ is odd.} \end{cases}$$

Definition 2.3. An h-magic graph G is said to be *uniformly null* if every h-magic labeling of G induces 0 sum.

Theorem 2.3. *The bipartite graph* $K(m,n)$ *is uniformly null if and only if* $|m-n| = 1$.

One can introduce a number of operations among zero-sum graphs which produce magic graphs. Here is an example of one such operation.

Definition 2.4. Given n graphs G_i $i = 1, 2 \cdots, n$, the chain $G_1 \diamond G_2 \diamond \cdots \diamond G_n$ is the graph in which one of the vertices of G_i is identified with one of the vertices of G_{i+1}. If $G_i = G$, we use the notation $\diamond G^n$ for the n-link chain all of whose links are G.

Observation 2.1. If graphs G_i have zero sum, so does the chain $G_1 \diamond G_2 \diamond \cdots \diamond G_n$, hence it is a magic graph. Moreover, if $G_i = G$, then the null set of the chain $\diamond G^n$ is the same as $N(G)$.

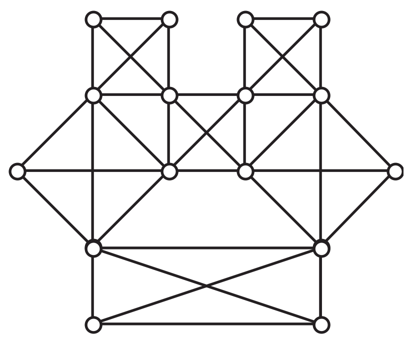

Figure 2.1. Finding the null set of this graph seems difficult.

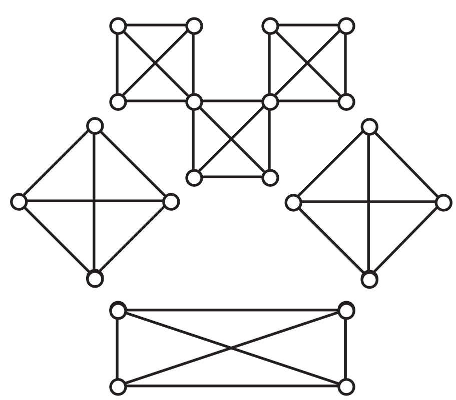

Figure 2.2. Graph G is constructed by six copies of K_4.

Theorem 2.4. $N(C_n) = \begin{cases} \mathbb{N} & \textit{if } n \textit{ is even} \\ 2\mathbb{N} & \textit{if } n \textit{ is odd} \end{cases}$

For any three positive integers $\alpha < \beta \leq \gamma$, the *theta graph* $\theta_{\alpha,\beta,\gamma}$ consists of three edge disjoint paths of length α, β and γ having the same endpoints, as illustrated in Figure 2.3. Theta graphs are also known as cycles with a P_k chord.

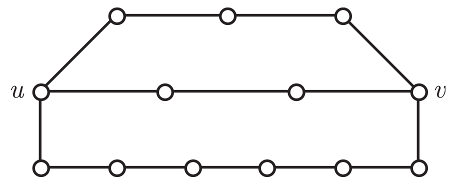

Figure 2.3. The graph $\theta_{3,4,7}$.

Theorem 2.5.

$$N(\theta_{\alpha,\beta,\gamma}) = \begin{cases} \mathbb{N} - \{2\} & \text{if } \alpha, \beta, \gamma \text{ have the same parity} \\ 2\mathbb{N} - \{2\} & \text{otherwise} \end{cases}$$

When k copies of C_n share a common edge, it will form an *n-gon book of k pages* and is denoted by $B(n, k)$.

Theorem 2.6.

$$N(B(n,k)) = \begin{cases} \mathbb{N} & n \textit{ is even, } k \textit{ is odd} \\ \mathbb{N} - \{2\} & n \textit{ and } k \textit{ are both even} \\ 2\mathbb{N} - \{2\} & n \textit{ is odd, } k \textit{ is even} \\ 2\mathbb{N} & n \textit{ and } k \textit{ both are odd} \end{cases}$$

Lemma 2.1. (Alternating label) Let u_1, u_2, u_3 and u_4 be four vertices of a graph G that are adjacent ($u_1 \sim u_2 \sim u_3 \sim u_4$) and $\deg u_2 = \deg u_3 = 2$. Then in any magic labeling of G the edges u_1u_2 and u_3u_4 have the same label.

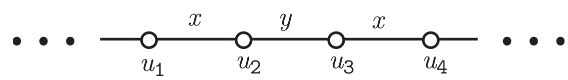

Figure 2.4. Alternating label in a magic labeling.

Given $k \geq 2$ the positive integers $a_1 < a_2 \leq a_3 \leq \cdots \leq a_k$, the generalized theta graph $\theta(a_1, a_2, \cdots, a_k)$ consists of k edge disjoint paths of lengths $a_1, a_2, \cdots, a_k$ having the same initial and terminal points.

When discussing magic labeling of a generalized theta graph $G = \theta(a_1, a_2, \cdots, a_k)$, the alternating label lemma (2.1), allows us to assume that $a_i = 2$ or 3. For convenience, we will use $\theta(2^m, 3^n)$ to denote the generalized theta graph which consists of m paths of even lengths and n paths of odd lengths.

Theorem 2.7. *Following the above notations, for any two non-negative integers m, n*

$$N(\theta(2^m, 3^n)) = \begin{cases} 2\mathbb{N} - \{1 - (-1)^{m+n}\} & \text{if } m = 1 \text{ or } n = 1; \\ \mathbb{N} - \{1 - (-1)^{m+n}\} & \text{otherwise.} \end{cases}$$

CHAPTER 3

NULL SETS OF CERTAIN PLANAR GRAPHS

3.1 Null sets of Wheels

For $n \geq 3$, wheels, denoted W_n, are defined to be $C_n + K_1$, where C_n is the cycle of order n. The integer-magic spectra of wheels are determined in [10].

Theorem 3.1. *If* $n \geq 3$, *then* $IM(W_n) = \mathbb{N} - \{1 + (-1)^n\}$.

In this section we determine the null sets of wheels. Since the degree set of the W_n is $\{3, n\}$, W_n cannot have a zero-sum magic labeling in $\mathbb{Z}_2$. Therefore, for any $n \geq 3,\ 2 \not\in N(W_n)$. Let $u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1$ be the vertices of the cycle C_n and u the center vertex of the wheel. In some cases, for convenience, we may use u_{n+1} for u_1 and u_{-1}, u_0 for u_{n-1}, u_n, respectively. The following observation will be useful in finding the null sets of wheels.

Observation 3.1. If $l : E(W_n) \to \mathbb{Z}_h$ $(h \neq 2)$ is a zero-sum magic labeling, then

$$2\Big(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) + l(u_nu_1)\Big) \equiv 0 \pmod{h}.$$

Proof. Let $l : E(W_n) \to \mathbb{Z}_h$ be the edge labeling that provides zero-sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all spokes is 0. Also, $l^*(u_k) = 0$ $(1 \leq k \leq n)$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} l^*(u_k) &= 2\sum l(u_iu_{i+1}) + l^*(u) \\ &= 2\Big(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) + l(u_nu_1)\Big) \equiv 0.\end{aligned}$$

□

Observation 3.2. For every $n \geq 3,\ 3 \in N(W_n)$ if and only if $n \equiv 0 \pmod 3$.

Proof. If $n \equiv 0 \pmod 3$, then we label all the edges of W_n by 1 and this provides a zero-sum in $\mathbb{Z}_3$. Now suppose $n \not\equiv 0 \pmod 3$ and let $l : E(W_n) \to \mathbb{Z}_3$ be any magic labeling of W_n with zero-sum. Then by Observation 3.6, the sum of the labels of the outer edges is 0. Since the outer edges cannot all be labeled 1 (or 2), two adjacent outer edges would have labels 1 and 2. This implies that the spoke adjacent to these two outer edges must have label 0, which is not an acceptable label. $\square$

Observation 3.3. If W_n is zero-sum h-magic, so is W_{kn} for every $k \in \mathbb{N}$.

Proof. Following the notations used above, let $u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1$ be the vertices of the cycle C_n and u the center vertex of W_n. For W_{kn}, let $v_1 \sim v_2 \sim \cdots \sim v_{kn} \sim v_1$ be the vertices of C_{kn} and v be its center vertex. Also, assume that $f : E(W_n) \to \mathbb{Z}_h$ is a magic labeling of W_n with 0 sum. Now define $g : E(W_{kn}) \to \mathbb{Z}_h$ by $g(vv_m) = f(uu_i)$ whenever $m \equiv i \pmod n$ and $g(v_m v_{m+1}) = f(u_i u_{i+1})$ whenever $m \equiv i \pmod n$. Then for the induced vertex labeling $g^* : V(W_{kn}) \to \mathbb{Z}_h$ we have $g^*(v) = kf^*(u) = 0$. Moreover, given any v_m let $m = qn + r$ $(0 \le r \le n-1)$. Then $g^*(v_m) = g(v_{m-1}v_m) + g(v_m v_{m+1}) + g(vv_m) = f(u_{r-1}u_r) + f(u_r u_{r+1}) + f(uu_r) = f^*(u_r) = 0$. Therefore, g is a magic labeling of W_{kn} with 0 sum. $\square$

Corollary 3.1. For any $n \ge 1$, $N(W_{3n}) = \mathbb{N} - \{2\}$.

Proof. Note that $W_3 \cong K_4$, for which we have $N(W_3) = \mathbb{N} - \{2\}$. Therefore, by 3.3, $N(W_{3n}) = \mathbb{N} - \{2\}$. $\square$

Lemma 3.1. For any $n \ge 3$, $\mathbb{N} - \{2, 3\} \subset N(W_n)$.

Proof. To prove the lemma we consider the following four cases:

Case 1. Suppose $n \equiv 0 \pmod 4$ or $n = 4p$ for some $p \in \mathbb{N}$.

A zero-sum magic labeling of W_4 is provided in Figure 3.1, which indicates that for every $h > 3$, the graph W_4 admits a zero-sum magic labeling in $\mathbb{Z}_h$, where $-2 \equiv 2^{-1}$. Therefore, by Observation 3.3, W_{4p} has a zero-sum magic labeling in $\mathbb{Z}_h$. That is $\mathbb{N} - \{2, 3\} \subset N(W_{4p})$.

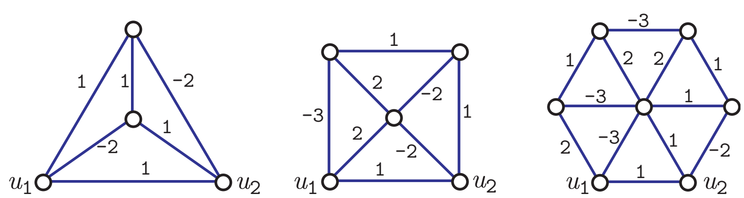

Figure 3.1. A zero-sum labeling of W_3, W_4 and W_6.

Case 2. Suppose $n \equiv 1 \pmod 4$ or $n = 4p + 1$ for some $p \in \mathbb{N}$. We proceed by induction on p and show that

> *"for any p, there is a zero-sum magic labeling for* W_{4p+1}*. Moreover, in this labeling at least one of the outer edges have label* 1*."*

Let $p = 1$. In Figure 3.2(A), a zero-sum magic labeling of W_5 in $\mathbb{Z}_4$ is provided. Also, Figure 3.2(B) indicates that W_5 admits a zero-sum magic labeling in $\mathbb{Z}_h$ for all $h \geq 5$, where $-1 \equiv 1^{-1}$, $-2 \equiv 2^{-1}$ and $-3 \equiv 3^{-1}$.

Now, assume that the statement is true for W_{4p+1} and let u_1u_2 be the outer edge of W_{4p+1} that has label 1. Then we eliminate this edge and insert the four-spoke extension, which is given in Figure 3.3, in such a way that the vertices z, v and w of this extension be identified with the central vertex u and vertices u_1, u_2 of W_{4p+1}, respectively. This provides the desired zero-sum magic labeling for W_{4p+5}.

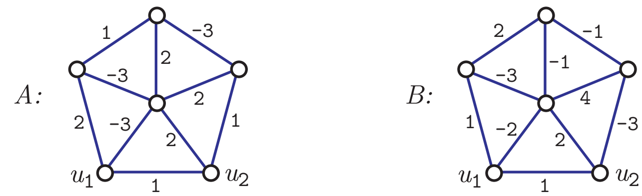

Figure 3.2. Two zero-sum-magic labeling of W_5.

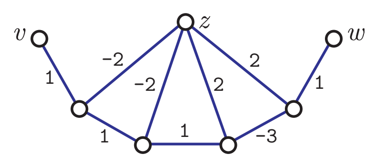

Figure 3.3. The four-spoke extension of a wheel.

An argument similar to the one presented in case 2, will also work for the remaining two cases:

Case 3. Suppose $n \equiv 2 \pmod 4$ or $n = 4p + 2$ for some $p \in \mathbb{N}$.

Case 4. Suppose $n \equiv 3 \pmod 4$ or $n = 4p + 3$ for some $p \in \mathbb{N}$. □

We summarize the main result of this section in the following theorem:

Theorem 3.2. *For any* $n \geq 3$, $N(W_n) = \begin{cases} \mathbb{N} - \{2\} & \text{if} n \equiv 0 \pmod 3 \\ \mathbb{N} - \{2, 3\} & \text{if } otherwise \end{cases}$

3.2 Null sets of Fans

For $n \geq 2$, Fans, denoted F_n, are defined to be $P_n + K_1$, where P_n is the path of order n. In this section we determine the null sets of Fans. Since the degree set of the F_n is

$\{2, 3, n\}$, it cannot have a magic labeling in $\mathbb{Z}_2$. Therefore, for any $n \geq 3$, $2 \not\in N(F_n)$. Note that $F_2 \equiv C_3$, and we know that $N(F_2) = 2\mathbb{N}$. Also, a typical magic labeling of $F_3 \cong K_4 - e$ is illustrated in Figure 3.4(A), for which we require that $a+b-z = a+b+z$ or $2z \equiv 0 \pmod{h}$; that is, h has to be even. On the other hand, if $h = 2r$, then F_3 admits a zero-sum magic labeling in $\mathbb{Z}_h$, as indicated in Figure 3.4(B). Therefore, $N(F_3) = 2\mathbb{N} - \{2\}$. For the general case, let $u_1 \sim u_2 \sim \cdots \sim u_n$ be the vertices of the path P_n and u the central vertex of the fan. We call the edges uu_i $(1 \leq i \leq n)$ blades of the fan F_n. The following observation will be useful in finding the null sets of fans.

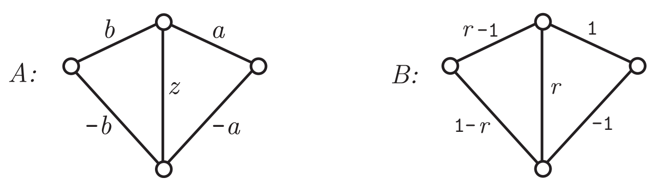

Figure 3.4. A typical magic labeling of F_3.

Observation 3.4. If $l : E(F_n) \to \mathbb{Z}_h$ $(h \neq 2)$ is a zero-sum magic labeling, then

$$2\Big(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\Big) \equiv 0 \pmod{h}.$$

Proof. Let $l : E(W_n) \to \mathbb{Z}_h$ be the edge labeling that provides zero-sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all blades is 0. Also, $l^*(u_k) = 0$ $(1 \leq k \leq n)$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} l^*(u_k) &= 2\sum l(u_iu_{i+1}) + l^*(v) \\ &= 2\Big(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\Big) \equiv 0.\end{aligned}$$

Theorem 3.3. $N(F_2) = 2\mathbb{N}$, $N(F_3) = 2\mathbb{N} - \{2\}$ *and for any* $n \geq 4$,

$$N(F_n) = \begin{cases} \mathbb{N} - \{2\} & \text{if } n \equiv 1 \pmod 3; \\ \mathbb{N} - \{2,3\} & \text{otherwise.} \end{cases}$$

Proof. First let us first consider if $n \geq 2,\ 3 \in N(F_n)$. We know from above that $3 \notin N(F_n)$ for $n = 2, 3$. Suppose $n \geq 4$ and $n \equiv 1 \pmod 3$. Then we label all the edges of P_n by 2, the two outer blades by 1 and all other blades by 2, as illustrated in Figure 3.5. This is a zero-sum magic labeling of F_n.

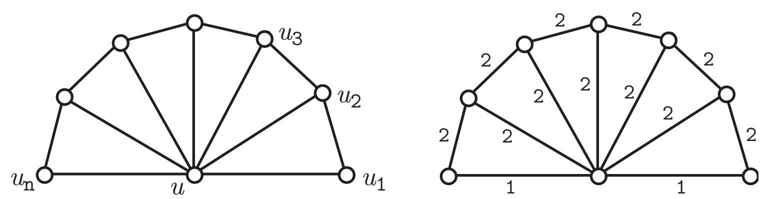

Figure 3.5. The fan F_n $(n = 7)$.

Next, suppose $n \not\equiv 1 \pmod 3$ and let $l : E(F_n) \to \mathbb{Z}_3$ be a zero-sum magic labeling. By Observation 3.4, we require that $l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) \equiv 0 \pmod 3$, which implies that at least two adjacent edges of P_n are labeled 1 and 2. But this will force the label of the blade adjacent to these edges be 0, which is not an acceptable label. Therefore, such a zero-sum magic labeling does not exist and $n \geq 2,\ 3 \in N(F_n)$ if and only if $n \equiv 1 \pmod 3$.

To finish the proof of the theorem we consider the following three cases:

Case 1. Suppose $n \equiv 1 \pmod 3$ or $n = 3p + 1$ for some $p \in \mathbb{N}$. We proceed by induction on p and show that

"for any p, there is a zero-sum magic labeling for F_{3p+1}. Moreover, in this

labeling at least one of the edges of P_n has label 2.*"*

Let $p = 1$. In Figure 3.6, a zero-sum magic labeling of F_4 is provided in $\mathbb{Z}_h$ for all $h \geq 4$, where $-1 \equiv 1^{-1}$, $-2 \equiv 2^{-1}$ and $-3 \equiv 3^{-1}$.

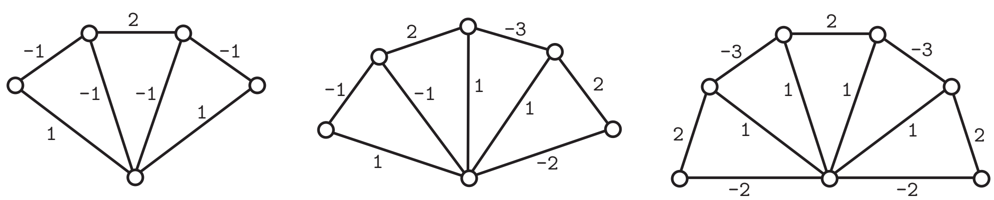

Figure 3.6. A zero-sum magic labelings of F_4, F_5 and F_6.

Now, assume that the statement is true for F_{3p+1} and let u_iu_{i+1} be the edge of P_{3p+1} that has label 2. Then we eliminate this edge and insert the three-blade extension, which is given in Figure 3.7, in such a way that the vertices z, v and w of this extension be identified with vertices u, u_i, u_{i+1} of F_{3p+1}, respectively. This provides the desired zero-sum magic labeling for F_{3p+4}.

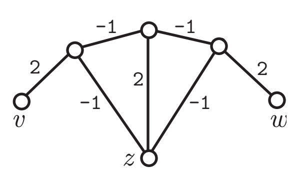

Figure 3.7. The three-blade extension of a fan.

An argument similar to the one presented in case 1, will also work for the remaining two cases:

Case 2. Suppose $n \equiv 2 \pmod 3$ or $n = 3p + 2$ for some $p \in \mathbb{N}$.

Case 3. Suppose $n \equiv 0 \pmod 3$ or $n = 3p$ for some $p \in \mathbb{N} - \{0\}$. □

3.3 Null sets of Double Wheels

For $n \geq 3$, double wheels, denoted by DW_n, are the cycle $C_n(u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1)$ together with two additional vertices v and w that are connected to all vertices of the cycle. In this section we determine the null sets of double wheels.

Theorem 3.4. *For all* $n \geq 3$ $N(DW_n) = \begin{cases} \mathbb{N} & \text{if } n \equiv 0 \pmod 2; \\ \mathbb{N} - \{2\} & \text{if } n \equiv 1 \pmod 2) \end{cases}$

Proof. To prove the Theorem we consider the following two cases:

Case 1. Suppose $n \equiv 1 \pmod 2$ or $n = 2p + 1$ for some $p \in \mathbb{N}$. First observe that if n is odd then the degree set of DW_n is $\{4, n\}$ and DW_n is not $\mathbb{Z}_2$-magic, therefore can not have a zero-sum labeling in $\mathbb{Z}_2$. Now we proceed by induction on p and show that

> *"for any p, there is a zero-sum magic labeling for* DW_{2p+1}*. Moreover, in this labeling at least one of the edges of* C_n *has label* 1*."*

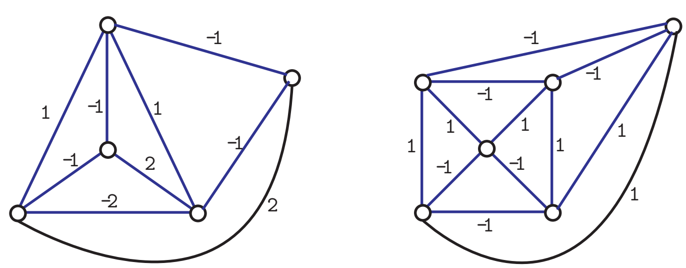

Figure 3.8. A zero-sum magic labeling of DW_3 and DW_4

In Figure 3.8, a zero-sum magic labeling of DW_3, $p = 1$, is provided in $\mathbb{Z}_h$ for all $h \geq 3$, where $-1 \equiv 1^{-1}$ and $-2 \equiv 2^{-1}$. Now, assume that the statement is true for DW_{2p+1} and let $u_i u_{i+1}$ be the edge of $C_{2p} + 1$ that has label 1. Then we eliminate this edge and insert the four-blade extension, given in Figure 3.9, with vertices a, b, c, and d such that a and d are identified with u_i and u_{i+1} respectively. This provides the desired zero-sum magic labeling for DW_{2p+3}.

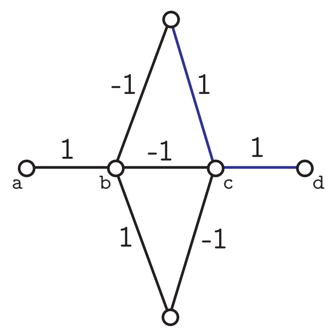

Figure 3.9. The four blade extension for a double wheel

Case 2. Suppose $n \equiv 0 \pmod 2$ or $n = 2p$ for some $p \in \mathbb{N} - \{0\}$. Then to prove $2 \in N(DF_n)$ one must observe that the degree set of $DW_{2p} = \{4, n\}$. Since n is even a labeling of 1 on all edges yields a zero-sum-magic labeling and $2 \in N(DW_n)$. Finally, an argument similar to one presented in case 1 will suffice to show that for $k \geq 3$, $k \in N(DW_n)$. □

3.4 Null sets of Double Fans

For $n \geq 2$, double fans, denoted by DF_n, are the path $C_n(u_1 \sim u_2 \sim \cdots \sim u_n)$ together with two additional vertices v and w that are connected to all vertices of the path. In this section we determine the null sets of Double Fans. Since the degree set of the DF_n is $\{3, 4, n\}$, it cannot have a magic labeling in $\mathbb{Z}_2$. Therefore, for any $n \geq 2$, $2 \not\in N(DF_n)$. Note that $DF_2 \equiv F_3$, and we know that $N(DF_2) = 2\mathbb{N} - \{2\}$. Also note, $DF_3 \equiv W_4$ and we know that $N(DF_3) = \mathbb{N} - \{2, 3\}$.

Theorem 3.5. $N(DF_2) = 2\mathbb{N} - \{2\}$, $N(DF_3) = \mathbb{N} - \{2, 3\}$, *and for any* $n \geq 4$,

$$N(DF_n) = \mathbb{N} - \{2\}$$

Proof. To prove the theorem we consider the following two cases:

Case 1. Suppose $n \equiv 0 \pmod 2$ or $n = 2p$ for some $p \in \mathbb{N} + 1 - \{0\}$. We proceed by induction on p and show that

> *"for any p, there is a zero-sum magic labeling for DF_{2p}. Moreover, in this labeling at least one of the edges of P_n has label* 2*."*

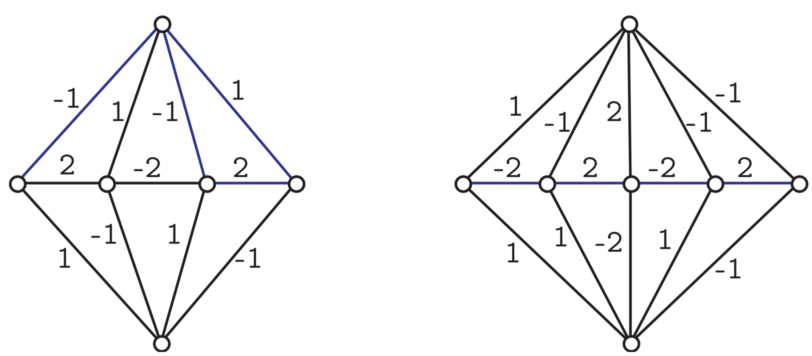

Figure 3.10. A zero-sum magic labeling of DF_4 and DF_5

In figure 3.10, a zero-sum magic labeling of DF_4, $p = 2$, is provided in $\mathbb{Z}_h$ for all $h \geq 3$, where $-1 \equiv 1^{-1}$ and $-2 \equiv 2^{-1}$. Now, assume that the statement is true for

DF_{2p} and let u_iu_{i+1} be the edge of P_{2p} that has label 2. Then we eliminate this edge and insert the four-blade extension, given in Figure 3.11, with vertices a, b, c, and d such that a and d are identified with u_i and u_{i+1} respectively. This provides the desired zero-sum magic labeling for DF_{2p+2}.

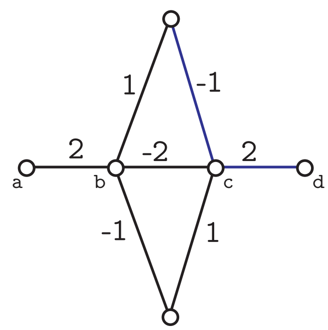

Figure 3.11. The four blade extension for a double fan

An argument similar to the one presented in case 1, will also work for the remaining case:

Case 2. Suppose $n \equiv 1 \pmod 2$ or $n = 2p+1$ for some $p \in \mathbb{N}+1$. $\square$

3.5 Prisms and n-Prisms

For $k \geq 3$ a Prism of order k, denoted P_2C_k, is $P_2 \times C_k$ In other words, two identical copies of C_k, $u_1 \sim u_2 \sim \cdots \sim u_k \sim u_1$ and $v_1 \sim v_2 \sim \cdots \sim v_k \sim v_1$, with additional edges connecting u_i and v_i for all i. For $k \geq 3$ and $n \geq 3$ an n-Prism of order k, is defined to be $P_nC_k = C_k \times P_n$. The degree set of P_2C_k is $\{3\}$ and the degree set of P_nC_k is $\{3, 4\}$. Therefore P_2C_k and P_nC_k cannot have a zero-sum magic labeling in $\mathbb{Z}_2$, which implies $2 \notin N(P_2C_k)$ and $2 \notin N(P_nC_k)$.

Theorem 3.6. *For any $k \geq 3$, $N(P_2C_k) = \mathbb{N} - \{2\}$*

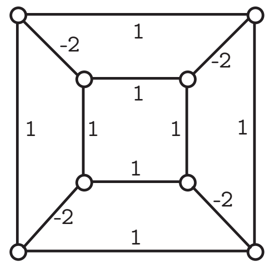

Figure 3.12. A zero-sum magic labeling P_2C_4

Proof. P_2C_k can be drawn as two cycles of order k, one within the other and oriented in the same way, with edges connected the corresponding vertices of the cycles. Label all of the edges of the cycles with 1, and all of the connecting edges with -2. Then all of the vertices have two edges labeled 1 and one edge labeled -2 incident. This is a zero-sum magic labeling of P_2C_k in $\mathbb{Z}_h$ for all $h \geq 3$, where -2 stands for the inverse of 2. □

Theorem 3.7. *For any $k \geq 3$ and $n \geq 3$, $N(P_nC_k) = \mathbb{N} - \{2\}$*

Proof. P_nC_k can be drawn as n cycles of order k, within each other and oriented in the same way, with edges connecting the corresponding vertices. Label all of the edges of the outermost cycles with 1, all the edges of the interior cycles with 2, and all of the connecting edges with -2. Then all of the vertices on the outermost cycles have two edges labeled 1 and one edge labeled -2 incident and all the vertices on the interior cycles have two edges labeled 2 and two edges labeled -2 incident. This is a zero-sum magic labeling of P_2C_k in $\mathbb{Z}_h$ for all $h \geq 3$. □

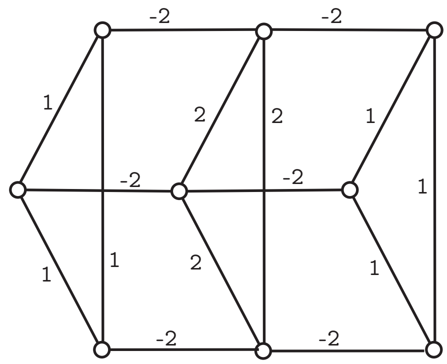

Figure 3.13. A zero-sum magic labeling P_3C_3

3.6 Anti-Prisms and n-Anti-Prisms

For $k \geq 3$ an anti-Prism of order k, denoted by AP_k, is two identical copies of C_k, C_k:$u_1 \sim u_2 \sim \cdots \sim u_k \sim u_1$ and $C_k^{'}$: $v_1 \sim v_2 \sim \cdots \sim v_k \sim v_1$ with additional edges u_iv_i and u_iv_{i-1} (mod n) for $i = 1 \cdots n$, as illustrated in Figure 3.14.

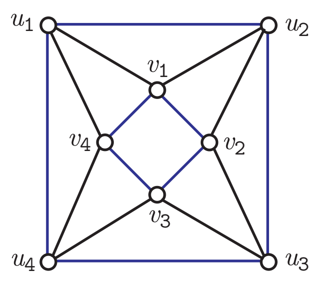

Figure 3.14. AP_4

For $k \geq 3$ and $n \geq 3$, an n-Anti-Prism of order k, n-AP_k, is n identical copies of C_k, $C_k^{(1)}$, $C_k^{(2)}$, $\cdots$, $C_k^{(n)}$, where any two consecutive cycles form an Anti-Prism.

Theorem 3.8. *For any $k \geq 3$, $N(AP_k) = \mathbb{N}$*

Proof. AP_k can be drawn in a planar fashion as two cycles of order k, one within the other with edges connected the corresponding vertices. If all the cycle edges are labeled with a 1 and all the edges connecting the two cycles are labeled with a -1, then all vertices with have two edges labeled with a 1 and two edges labeled with a -1. This is a zero-sum magic labeling of AP_k in $\mathbb{Z}_h$ for all $h \geq 2$, where -1 stands for the inverse of 1. □

Theorem 3.9. *For any $k \geq 3$ and $n \geq, 3$ $N(n\text{-}AP_k) = \mathbb{N}$*

Proof. n-AP_k can be drawn in a planar fashion as n cycles of order k, drawn one within another with edges connected the corresponding vertices. Label the edges in the following fashion: since all of the vertices are either degree 4 or degree 6 if we label ever edges with a 1 that is clearly a zero-sum labeling in $\mathbb{Z}_2$. Also if we label the outermost cycle edges with 1's, all of the inner cycle edges with 2's and all edges connecting cycles with -1's, then every vertex on the outermost cycles will have two edges labeled with 1 and two edges labeled with -1 and all inner cycle vertices will have two edges labeled with 2 and four edges labeled with -1. This is a zero-sum magic labeling of n-AP_k in $\mathbb{Z}_h$ for all $h \geq 3$. □

3.7 Null sets of Grids

For $n \geq 2$ and $k \geq 2$, an n by k Grid, $P_{n,k}$, is $P_n \times P_k$. In this section we determine the null sets of Grids. Since the degree set of $G_{n,k}$, for $n \geq 3$ and $k \geq 3$, is $\{2, 3, 4\}$ and the degree set of $G_{n,2}$, $n \geq 3$, and $G_{2,k}$, $k \geq 3$, is $\{2, 3\}$, $P_{n,k}$, where $n \geq 3$ or $k \geq 3$, cannot have a magic labeling in $\mathbb{Z}_2$ if $n \geq 3$ or $k \geq 3$. Therefore $2 \not\in N(P_{n,k})$ if $n \geq 3$ or $k \geq 3$. Note that $P_{2,2} \equiv C_4$, and we therefore know that $N(P_{2,2}) = \mathbb{N}$.

Theorem 3.10. *For any $n \geq 3$ or $k \geq 3$, $\mathbb{N} - \{2\} \subset N(P_{n,k})$*

Proof. You can think of a Grid as being k P_n's oriented horizontally and connected together by n P_k's. Label all of the edges in the following way: the two outer P_n's with 1's, the $k-2$ inner P_n's with 2's, the two outer P_{k+1}'s with -1's, and the the $n-2$ outer P_k's with -2's.

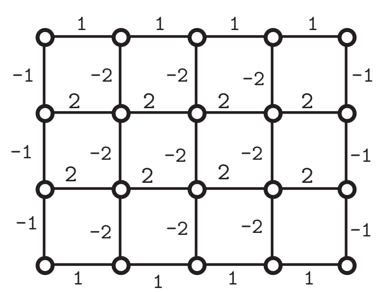

Figure 3.15. A zero-sum magic labeling of $P_{4,5}$

Then the corner vertices will have edges incident to them having values 1 and -1, the non-corner vertices on the outer P_n's will have edges incident to them having values 1, 1, and -2, the non-corner vertices on the outer P_k's will have edges incident to them having values -1, -1, and 2, and all interior vertices will have two edges incident to them having value 2 and two others having value -2. This is a zero-sum magic labeling of $G_{n,k}$ in $\mathbb{Z}_h$ for all $h \geq 3$, where -1 and -2 stand for the inverses of 1 and 2 respectively. □

3.8 Null sets of Bowties

For $n \geq 2$, Bowties, BT_n, are two identical copies of F_n that are connected together at the F_ns' K_1 nodes. Since the degree set of the BT_n is $\{2, 3, n+1\}$, BT_n cannot have a zero-sum magic labeling in $\mathbb{Z}_2$. Therefore, for any $n \geq 2$, $2 \not\in N(BT_n)$. Let $u_1 \sim u_2 \sim \cdots \sim u_n$ be the vertices of the path of one of the F_ns, u the K_1 vertex of the corresponding fan, and v the K_1 of the other fan. The following observation will be useful in finding the null sets of bowties.

Observation 3.5. If $l : E(BT_n) \to \mathbb{Z}_h$ is a zero-sum magic labeling, then

$$2\bigg(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\bigg) \equiv l(uv) \pmod{h}.$$

Proof. Let $l : E(BT_n) \to \mathbb{Z}_h$ be the edge labeling that provides zero-sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all spokes is $-l(uv)$. Also, $l^*(u_k) = 0$ $(1 \leq k \leq n)$ and $l^*(u) = 0$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} l^*(u_k) &= 2\sum_{k=1}^{n} l(u_iu_{i+1}) + \bigg(l^*(u) - l(uv)\bigg) \\ &= 2\sum_{k=1}^{n} l(u_iu_{i+1}) - l^*(uv) \\ &\Rightarrow 2\bigg(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\bigg) \equiv l(uv).\end{aligned}$$

□

Theorem 3.11. *For any $n \geq 2$,*

$$N(BT_n) = \begin{cases} \mathbb{N} - \{2,3\} & \text{if } n \equiv 1 \pmod 3; \\ \mathbb{N} - \{2\} & \text{otherwise.} \end{cases}$$

Proof. First let us consider if $n \geq 2,\ 3 \not\in N(BT_n)$. For $n \not\equiv 1 \pmod 3$ label all the path edges with 1's, all the interior spokes with 1's, all the exterior spokes with 2's, and the uv edge with a 1 if $n \equiv 0 \pmod 3$ or 2 if $n \equiv 2 \pmod 3$. Then the end vertices of the path have an edge labeled 1 and an edge labeled 2 incident, the interior path vertices have three edges labeled 1 incident, and if $n \equiv 0 \pmod 3$ u and v have two edges labeled 2, $n-1$ edges labeled 1 incident or if $n \equiv 2 \pmod 3$ u and v have three edges labeled 2 and $n-2$ edges labeled 1 incident. These are both clearly zero-sum labeling. Suppose $n \equiv 1 \pmod 3$ and let $l : E(BT_n) \to \mathbb{Z}_3$ be any magic labeling of BT_n with zero-sum. Then by Observation 3.5, twice the sum of the labels of the path edges is $l(uv)$. Since there are $n-1$ path edges, which is equivalent to $0 \pmod 3$, they cannot all be labeled 1 (or 2) since that would imply that $l(uv)$ is zero, two adjacent path edges would have labels 1 and 2. This implies that the spoke adjacent to these two outer edges must have label 0, which is not an acceptable label. Therefore, $n \geq 2,\ 3 \not\in N(BT_n)$ if and only if $n \equiv 1 \pmod 3$.

To finish the proof of the theorem we consider the following three cases:

Case 1. Suppose $n \equiv 1 \pmod 3$ or $n = 3p+1$ for some $p \in \mathbb{N}$. We proceed by induction on p and show that

> *"for any p, there is a zero-sum magic labeling for BT_{3p+1}. Moreover, in this labeling at least one of the edges on both P_n's has label* 1.*"*

In figure 3.16, a zero-sum magic labeling of BT_4, $p = 1$, is provided in $\mathbb{Z}_h$ for all $h \geq 4$, where $-2 \equiv 2^{-1}$ and $-3 \equiv 3^{-1}$.

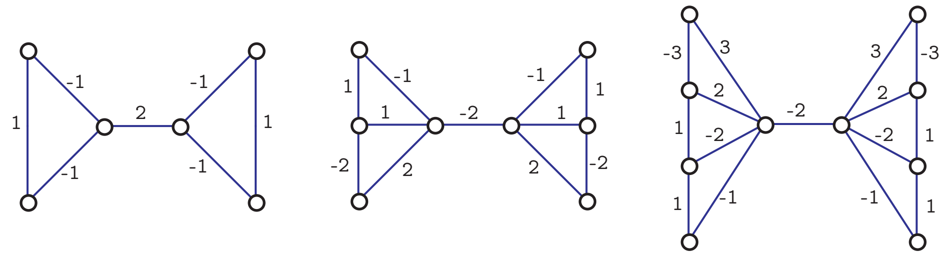

Figure 3.16. A zero-sum magic labeling BT_2, BT_3, and BT_4

Now, assume that the statement is true for BT_{3p+1} and let $u_i u_{i+1}$ be the edge of P_{3p+1} that has label 1. Then we eliminate this edge and insert the four-blade extension, given in Figure 3.17, with vertices a, b, c, d, and e, such that a and e are identified with u_i and u_{i+1} respectively. This provides the desired zero-sum magic labeling for BT_{3p+4}.

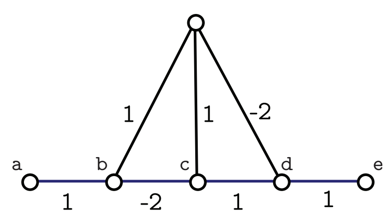

Figure 3.17. The three blade extension of a bowtie.

An argument similar to the one presented in case 1, will also work for the remaining two cases:

Case 2. Suppose $n \equiv 2 \pmod 3$ or $n = 3p + 2$ for some $p \in \mathbb{N}$.

Case 3. Suppose $n \equiv 0 \pmod 3$ or $n = 3p$ for some $p \in \mathbb{N} - \{0\}$. □

3.9 Null sets of Axles

For $n \geq 3$, Axles, AX_n, are two identical copies of W_n that are connected together at the W_ns' K_1 nodes. Since the degree set of the AX_n is $\{3, n+1\}$, AX_n cannot have a zero-sum magic labeling in $\mathbb{Z}_2$. Therefore, for any $n \geq 2$, $2 \not\in N(AX_n)$. Let $u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1$ be the vertices of the cycle of one of the W_n's(in some cases, for convenience, we may use u_{n+1} for u_1 and u_{-1}, u_0 for u_{n-1}, n_n, respectively.), u the K_1 vertex of the corresponding wheel, and v the K_1 of the other wheel.

Observation 3.6. If $l : E(AX_n) \to \mathbb{Z}_h$ is a zero-sum magic labeling, then

$$2\Bigg(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n + l(u_nu_1)\Bigg) \equiv l(uv) \pmod{h}.$$

Proof. Let $l : E(AX_n) \to \mathbb{Z}_h$ be the edge labeling that provides zero-sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all spokes is $-l(uv)$. Also, $l^*(u_k) = 0$ $(1 \leq k \leq n)$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} l^*(u_k) &= 2\sum_{k=1}^{n} l(u_iu_{i+1}) + \Big(l^*(u) - l(uv)\Big) \\ &= 2\sum_{k=1}^{n} l(u_iu_{i+1}) - l^*(uv) \\ &\Rightarrow 2\Bigg(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\Bigg) \equiv l(uv).\end{aligned}$$

□

Theorem 3.12. *For any $n \geq 3$,*

$$N(AX_n) = \begin{cases} \mathbb{N} - \{2,3\} & \text{if } n \equiv 0 \pmod 3 \\ \mathbb{N} - \{2\} & \text{otherwise} \end{cases}$$

Proof. First let us consider if $n \geq 3$, $3 \not\in N(AX_n)$. For $n \not\equiv 0 \pmod 3$ label all the cycle and spoke edges with 1's and the edge uv with a 1 if $n \equiv 2 \pmod 3$ or 2 if $n \equiv 1 \pmod 3$. Then the cycle vertices of the wheel have three edges labeled 1 incident and

u and v have n edges labeled 1 and one edge labeled 2 incident if $n \equiv 1 \pmod 3$ or $n + 1$ edges labeled 1 if $n \equiv 2 \pmod 3$. These are both clearly zero-sum labelings. Suppose $n \equiv 1 \pmod 3$ and let $l : E(AX_n) \to \mathbb{Z}_3$ be any magic labeling of BT_n with zero-sum. Then by Observation 3.6, twice the sum of the labels of the cycle edges is $l(uv)$. Since there are n cycle edges, which is equivalent to $0 \pmod 3$, they cannot all be labeled 1 (or 2) since that would imply that $l(uv)$ is zero, two adjacent path edges would have labels 1 and 2. This implies that the spoke adjacent to these two outer edges must have label 0, which is not an acceptable label. Therefore, $n \geq 3,\ 3 \notin N(AX_n)$ if and only if $n \equiv 0 \pmod 3$.

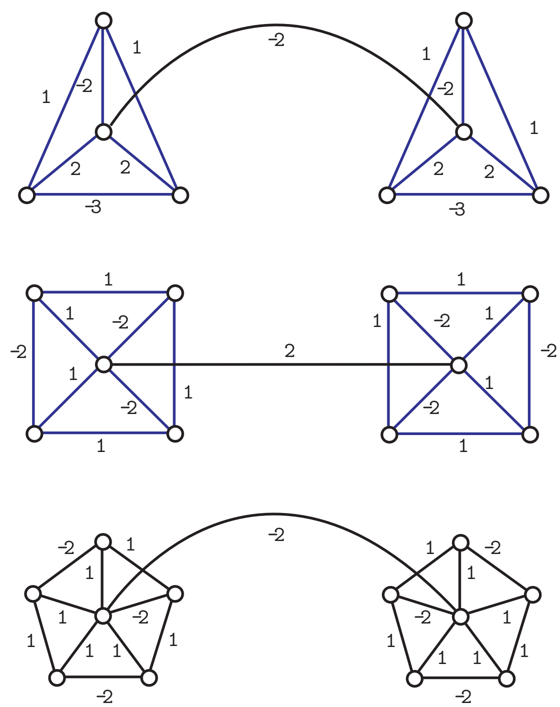

Figure 3.18. A zero-sum magic labeling AX_3, AX_4, and AX_5

To finish the proof of the theorem we consider the following three cases:

Case 1. Suppose $n \equiv 0 \pmod 3$ or $n = 3p$ for some $p \in \mathbb{N}$. We proceed by induction on p and show that

> *"for any p, there is a zero-sum magic labeling for AX_{3p}. Moreover, in this labeling at least one of the edges on both W_n's has label 1."*

In figure 3.18, a zero-sum magic labeling of AX_3, $p = 1$, is provided in $\mathbb{Z}_h$ for all $h \geq 4$, where $-2 \equiv 2^{-1}$ and $-3 \equiv 3^{-1}$.

Now, assume that the statement is true for AX_{3p} and let $u_i u_{i+1}$ be the edge of AX_{3p} that has label 1. Then we eliminate this edge and insert the four-blade extension, given in Figure 3.19, with vertices a, b, c, d, and e, such that a and e are identified with u_i and u_{i+1} respectively.. This provides the desired zero-sum magic labeling for AX_{3p+4}.

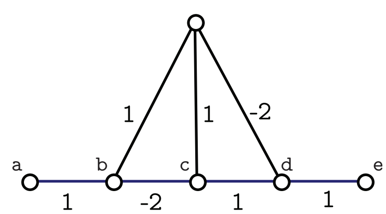

Figure 3.19. The three blade extension of an axle.

An argument similar to the one presented in case 1, will also work for the remaining two cases:

Case 2. Suppose $n \equiv 1 \pmod 3$ or $n = 3p + 1$ for some $p \in \mathbb{N}$.

Case 3. Suppose $n \equiv 2 \pmod 3$ or $n = 3p + 2$ for some $p \in \mathbb{N} - \{0\}$. $\square$

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Thesis Title:

Zero-Sum Magic Graphs and Their Null Sets

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